TOPOLOGY IN B-METRIC SPACES

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CHAPTER 1

INTRODUCTION

In his thesis (Sur quelques points du calcul fonctionnel (1906)) Maurice Fréchet introduced the notion of an abstract space bearing an écart. Since that time semimetric and metric spaces have played the central role in topology and modern analysis. Around 1928 the study of the geometry of such spaces was begun by Karl Menger and many others. In recent years, the notions which have proved so fruitful in the classical case have been extended, in large part, to spaces with more abstract distance functions (1, 2, 5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 18, 23, 24).[1] More recently still the foundations for a general theory of distance spaces have been given (12). The purpose of this dissertation is to begin the development of the primarily topological part of the study of "generalized metric spaces" (12) whose ground sets are σ-complete Boolean algebras, the geometry of such spaces having been previously considered (5, 10, 11).

Two of the more interesting results obtained are the completion of these B-metric spaces and an application to

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[1]Numerals in parentheses refer to the bibliography concluding the dissertaion.

general topology wherein it is shown that any zero-dimensional space may be considered a B-metric space.

It should be noted that the major tool used in this dissertation, namely, the solution of Lattice Theory Problem 77 (3), became available only within the past year (13).

CHAPTER II

PRELIMINARY NOTIONS

In this chapter will be introduced some notions necessary for later arguments.

1. Lattice Theory

A lattice is a set of elements closed under two binary operations ($\wedge, \vee$) which are defined for all pairs of elements and which satisfy the following postulates:

(1) $x \wedge x = x = x \vee x$, for all x (idempotency).

(2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee x) = (x \vee y) \vee x$, for all x, y, z (associativity).

(3) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$, for all x, y (commutativity).

(4) $x \vee y = y$ if and only if $x \wedge y = x$, for all x, y (alternation). If the lattice also satisfies:

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for all x, y, z, then the lattice is said to be distributive.

It is possible to define a lattice as a set of elements with one binary relation ($\leq$) which satisfies the following postulates:

(1) $x \leq x$, for all x (reflexivity).

(2) $x \leq y$ and $y \leq x$ implies $x = y$, for all x, y (antisymmetry).

(3) $x \leq y$ and $y \leq z$ implies $x \leq z$; for all x, y, z (transitivity)

(4) For any x, y there exists an element z, called the greatest lower bound (g.l.b.), such that $z \leq x$ and $z \leq y$ and if $a \leq x$ and $a \leq y$, then $a \leq z$.

(5) For any x, y there exists an element z, called the least upper bound (l.u.b.), such that $x \leq z$, $y \leq z$, and if $x \leq a$ and $y \leq a$, then $z \leq a$.

These two sets of conditions can be shown to be equipollent if one defines $x \leq y$ if and only if $x \wedge y = x$ to get the second set from the first, and if one defines $x \wedge y$ as the g.l.b. of x, y and $x \vee y$ as the l.u.b. of x, y to get the first set from the second.

If there are two elements 0, 1 of the lattice which satisfy:

(6) $0 \leq x$, for all x.

(7) $x \leq 1$, for all x.

then the lattice is said to have a first element and a last element. If further a lattice (with 0, 1) satisfies

(8) for any x there exists an element x' such that $x \wedge x' = 0$ and $x \vee x' = 1$, then the lattice is said to be complemented.

A Boolean algebra is a distributive complemented lattice.

An element in a Boolean algebra which follows each element of a countably infinite set and which precedes any element which follows all is called the l.u.b. of the set and is denoted by $\bigvee_{i=1}^{\infty} x_i$. Also an element in a Boolean algebra which precedes each element of a countably infinite set and which follows any element which precedes all is called the g.l.b. of the set and is denoted by $\bigwedge_{i=1}^{\infty} x_i$. If each countable set has a l.u.b. and a g.l.b. then the Boolean algebra is said to be a σ-complete Boolean algebra.

In a σ-complete Boolean algebra $\limsup x_i = \overline{\lim_i}\, x_i = \bigwedge_{j=1}^{\infty} \bigvee_{i=j}^{\infty} x_i$ and $\liminf x_i = \underline{\lim_i}\, x_i = \bigvee_{j=1}^{\infty} \bigwedge_{i=j}^{\infty} x_i$ each exist since the l.u.b. and the g.l.b. each exist for any countable set of elements. If the $\overline{\lim_i}\, x_i = \underline{\lim_i}\, x_i$, then the limit is said to exist and is $\lim_i x_i$.

If all subsets (including the null (0) set) of a given set of cardinal $\aleph$ are considered and set inclusion is the binary relation, then the resulting structure is called a full set algebra and has $2^{\aleph}$ elements. This full set algebra can be shown to be a σ-complete Boolean algebra.

For greater detail consult G. Birkhoff (3).

2. General Topology

A set of elements with an undefined notion ($^{-}$, closure) is said to be a Kuratowski space if it satisfies the following postulates:

(1) For every set X, of elements, there is a closure, $\overline{X}$.

(2) $X \subset \overline{X}$, for all X.

(3) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, for all X, Y.

(4) $\overline{\overline{X}} \subset \overline{X}$, for all X.

(5) $\overline{0} = 0$ (where 0 is the null set).

A set X is closed if $\overline{X} = X$.

A set is called an open set if the set of elements which are not in it (its complement) is closed.

A point or element, p, is said to be an accumulation point of a set if every open set which has p as an element contains an element of the set distinct from p. A set is closed if it contains all of its accumulation points. A closed set defined in terms of accumulation points is a closed set defined in terms of closure, and conversely.

If a sequence is defined as a mapping of the integers into a set then the $\lim_i x_i = x$ (where the $\{x_i\}$ are a sequence) providing any open set containing x contains almost all of the x_i (all but a finite number).

A mapping, f, of one space onto another (f: $K \to L$) is a continuous mapping providing for each open set, U, in L there exists an open set, V, in K so that $f(V) = U$.

Had the definition of a Kuratowski space been given in terms of limits of sequences, then the two systems could be shown to be equipollent. The closure of a set is gotten by taking limits of all sequences in the set and including them in the set. It is unnecessary to take limits of limits of sequences as no new elements will result; that is, we get closure by one application of the limit function (25).

A topological space is said to be a T_0-space if for any elements x, y of the space there is an open set about one which does not contain the other.

A topological space is said to be a T_1-space if for any elements x, y of the space there is an open set about each which does not contain the other. An equivalent condition is that all finite sets be closed.

A topological space is said to be a T_2-space (Hausdorff) if for any elements x, y of the space there is an open set, U, about x and an open set, V, about y and $U \cap V = 0$. An equivalent condition in a T_1-space is that limits be unique.

A topological space which is T_1 is said to be a T_3-space (regular) if for any closed set X and a point, p, not in it there exist two open sets U, V, $X \subset U$, $p \in V$, and $U \cap V = 0$.

A topological space which is T_1 is said to be a T_4-space (normal) if for any two disjoint closed sets, X, Y, (i.e. $X \cap Y = 0$) there exist two open sets U, V with $X \subset U$,

$Y \subset V$, and $U \cap V = 0$.

A topological space is said to be compact if every infinite set has a point of accumulation.

If every point of the space is a point of accumulation of the space then the space is dense in itself.

If, in a Kuratowski space X, for any closed set A, and any element p not in A, there exists a continuous mapping f, of X onto a subset of the interval $[0, 1]$ such that $f(A) = 0$ and $f(p) = 1$, then X is said to be completely regular.

For greater detail consult R. Vaidyanathaswamy (25).

3. Distance Spaces.

A set of elements with a function (d) on $SS \to R$ is called a metric space providing it satisfies the following postulates.

(1) $d(x, y) = 0$ if and only if $x = y$; for all x, y (vanishing condition).

(2) $d(x, y) = d(y, x)$; for all x, y (symmetry).

(3) $d(x, y) + d(y, z) \geq d(x, z)$; for all x, y, z (triangle inequality).

It is possible to generalize this action so that the ground set (R in the above formulation) is any lattice. Such a space is called a "generalized metric space". In a metric space the concept of limit is defined as follows: $\lim_i x_i = x$ (where $\{x_i\}$ is a sequence) providing for any $\varepsilon > 0$ there is a positive integer N such that for any $n > N$, $d(x_n, x) < \varepsilon$.

A sequence $\{x_i\}$ is said to be a Cauchy sequence if for any $\varepsilon > 0$ there is a positive integer N such that for any $n, m > N$, $d(x_n, x_m) < \varepsilon$. It has been shown that any metric space can be completed (20); that is, a metric space can be densely congruently imbedded in a metric space which has limits for all Cauchy sequences. The question then becomes, when can a "generalized metric space" be completed (after satisfactory definitions of limit of a sequence and Cauchy

sequence have been given)?

For greater detail consult David Ellis (12).

Further notions will be developed as needed.

CHAPTER III

1. Preliminaries

In the remainder of the work, B shall always denote a σ-complete Boolean algebra. In B we denote the operations of join, complement, meet, and symmetric difference by $\vee$, $'$, $\wedge$, and $\oplus$, respectively. Symmetric difference can be defined in terms of $\vee$, $'$, and $\wedge$ as follows:

(a) $x \oplus y = (x \wedge y') \vee (x' \wedge y)$.

If $\{x_i\}$ is a sequence of points in B, then the limit of a sequence as defined in Chapter 2, Section 1 induces a topology in B known as the K-topology (Kantorovitch topology) or as the sequential order topology (3, 16, 21).

We shall need the following well-known and easily shown facts

(1) $\underline{\lim}_i\, x_i \leq \overline{\lim}_i\, x_i$, for arbitrary sequences (3).

(2) $\underline{\lim}_i\, (x_i \vee y_i) \leq \underline{\lim}_i\, x_i \vee \overline{\lim}_i\, y_i$, for arbitrary sequences (22).

(3) $\lim_i\, x_i = 0$ is equivalent to $\overline{\lim}_i\, x_i = 0$ (this follows immediately from (1)), for arbitrary sequences.

(4) The Boolean operations ($\vee$, $'$, $\wedge$, $\oplus$) are continuous in the K-topology of B (3, 22).

We shall also need the following Lemma obtained by David Ellis (13).

<u>Lemma 1</u>. <u>If, in</u> S, $\lim_j x_j^i = x_i$, <u>for all</u> i, <u>and</u> $\lim_i x_i = x$, <u>then there is a function</u> j(i) <u>so that</u> $k(i) \geq j(i)$; <u>for all</u> i, <u>implies</u> $\lim_i x_{k(i)}^i = x$.

By a S-metric space we shall mean a set S together with a function (d) which satisfies the three conditions in Chapter 2, Section 3 with "R" replaced by "S" and "+" replaced by "$\vee$". In the language of the paper (12), a S-metric space is a "generalized metric space over S".

If S is a S-metric space we define $d\text{-}\lim_i x_i = x$ in S if and only if $\lim_i d(x_i, x) = 0$ in S. The resulting sequential topology is called the d-topology or metric topology of S.

A topological space S is said to be metrizable over S if there is a function $d(x, y): SS \to S$, under which S forms a S-metric space such that $\lim_i x_i = x$ in the original topology of S if and only if $d\text{-}\lim_i x_i = x$.

It is to be recalled (10, 11) that S itself forms a S-metric space under the autometrization $d(x, y) = x \oplus y$.

2. <u>The autometrization of</u> S.

<u>Theorem 1</u>. <u>S in its K-topology is metrizable over itself by the autometrization</u>.

<u>Proof</u>. Since S is a S-metric space under $d(x, y) = x \oplus y$, we need only show that $\lim_i x_i = x$ if and only if

$\lim_i d(x, x_i) = \lim_i (x \oplus x_i) = 0$. The "only if" part is immediate from the continuity of $\oplus$ in the K-topology and the nilpotency ($x \oplus x = 0$) of $\oplus$. Suppose that $\lim_i (x \oplus x_i) = 0$, then $\lim_i ((x' \wedge x_i) \vee (x \wedge x_i') \vee (x \wedge x_i')) = 0$. $\overline{\lim_i} (x' \wedge x_i)$, $\overline{\lim_i} (x \wedge x_i') \leq \lim_i ((x' \wedge x_i) \vee (x \wedge x_i')) = 0$. Therefore $\overline{\lim_i} x_i \leq x$ and $x \leq \underline{\lim_i} x_i$. By transitivity we get the desired conclusion.

Theorem 1 implies that we shall obtain results about the K-topology of B as special cases of results dealing with general B-metric spaces. Some of these special cases have been previously obtained by Loewig (22), although the above definitions do not all agree with his.

3. <u>Continuity of the distance function</u>. <u>The closure in</u> B.

Let E be a B-metric space. It is well-known that limits are unique, subsequences of a convergent sequence converge to the same limit, and a sequence almost all of whose elements coincide converges to that element in the K-topology of B (3, 16, 21). These imply the corresponding facts in E. Thus the d-topology of E makes E a Hausdorff space.

<u>Theorem 2</u>. <u>If, in</u> E, $d\text{-}\lim_i x_i = x$ <u>and</u> $d\text{-}\lim_i y_i = y$, <u>then</u> $\lim_i d(x_i, y_i) = d(x, y)$.

<u>Proof</u>. Assume the hypothesis. By a repeated application of the triangle inequality,

(1) $d(x, y) \leq d(x, x_i) \vee d(x_i, y_i) \vee d(y, y_i)$, for all i.

Thus,

(2) $d(x, y) \leq \underline{\lim}_i (d(x, x_i) \vee d(x_i, y_i) \vee d(y, y_i)) \leq \underline{\lim}_i (d(x, x_i) \vee d(x_i, y_i)) \vee \overline{\lim}_i d(y, y_i) \leq \underline{\lim}_i d(x_i, y_i) \vee \overline{\lim}_i d(x, x_i) = \underline{\lim}_i d(x_i, y_i)$.

Similarly,

(3) $\overline{\lim}_i d(x_i, y_i) \leq d(x, y)$ from the fact that $d(x_i, y_i) \leq d(x_i, x) \vee d(y_i, y) \vee d(x, y)$.

(2) and (3) together yield Theorem 2. Theorem 2 may be restated. <u>The distance function of a B-metric space is simultaneously continuous</u>.

<u>Theorem 3</u>. <u>If, in B, d-lim</u> $x_j^i = x_i$ <u>for all</u> i, <u>and d-lim</u> $x_i = x$, <u>there is a function</u> j(i) <u>so that</u> $k(i) \geq j(i)$, <u>for all</u> i, <u>implies d-lim</u> $x_{k(i)}^i = x$.

<u>Proof</u>. Application of Theorem 2 to the hypotheses of Theorem 3 yields $\lim_i d(x_j^i, x) = d(x_i, x)$, for all i, and $\lim_i d(x_i, x) = 0$. Application of Lemma 1 now yields the conclusion of Theorem 3.

Since B is Hausdorff and forms a topological group under $\oplus$ in its N-topology, we may infer (19) that B is completely regular in its K-topology. This yields

<u>Theorem 4</u>. <u>Every B-metric space is completely regular in its d-topology</u>.

Proof. Let S be a B-metric space, and let C be a closed subset of S and p a point of S not in C. Since $d(p, x)$ is a continuous function of x in S and d-limits are unique, $F = \overline{\{d(p, x) \mid x \in C\}}$ is a closed set in S not containing 0 (if F contained 0 then we could construct a sequence of x_i's which would approach p and hence p would be an element of C, which would be a contradiction). Thus, there is a continuous mapping $f: S \to [0, 1]$ with $f(0) = 0$ and $f(x) = 1$; for x in F. Then $f(d(p, x))$ is a continuous mapping of S into $[0, 1]$ (the transformation product of two continuous functions being continuous) with $f(d(p, p)) = 0$ and $f(d(p, x)) = 1$; for $x \in C$.

Summarizing the results of this section we state:

Theorem 5. Let S be a B-metric space. S is a completely regular Hausdorff space in which the closure of a set is given by the single application of the tier function (25). (The tier function is a function which joins to a set its limit points).

Corollary. Let S be a B-metric space, $p \in S$, $C \subset S$. Then $p \in \overline{C}$ if and only if p is the d-limit of a sequence of points of C.

4. Cauchy sequences and the completion.

Throughout this section, S denotes a B-metric space. We define a Cauchy sequence in S as a sequence $\{x_i\}$ in S with the property:

(c) $\lim_i \lim_j d(x_i, x_j) = 0$.

One should note that this is weaker than Loewig's (22) definition of Cauchy sequence which requires the existence of an independent double limit. One should also note that (C) is equivalent to $\overline{\lim_i} \lim_j d(x_i, x_j) = 0$.

S is complete provided every Cauchy sequence in S converges to a point of S.

<u>Theorem 6</u>. <u>S is complete in the autometrization</u>

<u>Proof</u>. Suppose $\{x_i\}$ is a Cauchy sequence in S.

Now,

(1) $d(x_i, \overline{\lim_j} x_j) = (x_i \wedge (\overline{\lim_j} x_j)') \vee (x_i' \wedge \overline{\lim_j} x_j) = (x_i \wedge \underline{\lim_j} x_j') \vee (x_i' \wedge \overline{\lim_j} x_j) = \underline{\lim_j} (x_i \wedge x_j') \vee \overline{\lim_j} (x_i' \wedge x_j) \leq \overline{\lim_j} (x_i \wedge x_j') \vee \overline{\lim_j} (x_i' \wedge x_j) \leq \overline{\lim_j} d(x_i, x_j) = \lim_j d(x_i, x_j)$; for each i.

Thus,

(2) $\overline{\lim_i} d(x_i, \overline{\lim_j} x_j) = 0$ and $d\text{-}\lim_i x_i = \overline{\lim_j} x_j$.

<u>Theorem 7</u>. <u>If, in S, $d\text{-}\lim_i x_i$ exists, then $\{x_i\}$ is a Cauchy sequence</u>.

<u>Proof</u>. Let $\lim_i d(x, x_i) = 0$. By Theorem 2, $\lim_j d(x_i, x_j) = d(x, x_i)$, for all i. Hence $\lim_i \lim_j d(x_i, x_j) = \lim_i d(x, x_i) = 0$, again by Theorem 2.

<u>Lemma 2</u>. <u>If, in S, $\{x_i\}$ is a Cauchy sequence and $y \in S$, then $\lim_i d(x_i, y)$ exists</u>.

<u>Proof</u>.

(1) $d(x_i, y) \leq d(x_i, x_j) \vee d(y, x_j)$, for all i, j.

(2) $\overline{\lim_i} d(x_i, y) \leq d(x_j, y) \vee \lim_i d(x_i, x_j)$; for all j.

(3) $\overline{\lim_i}\, d(x_i, y) \leq \underline{\lim_j}\, (d(x_j, y) \vee \lim_i d(x_i, y_j)) \leq$
$\underline{\lim_j}\, d(x_j, y) \vee \overline{\lim_j}\, \lim_i d(x_i, x_j)$.
Thus,

(4) $\overline{\lim_i}\, d(x_i, y) \leq \underline{\lim_j}\, d(x_j, y)$, which concludes the proof.

<u>Lemma 3</u>. <u>If $\{x_i\}$ and $\{y_i\}$ are Cauchy sequences in S, then</u> $\lim_i d(x_i, y_i)$ <u>exists</u>.

<u>Proof</u>. Lemma 2 will be applied in this proof without comment.

(1) $d(x_i, y_i) \leq d(x_i, x_j) \vee d(x_j, y_i)$, for all i, j.

(2) $\overline{\lim_i}\, d(x_i, y_i) \leq \lim_i d(x_i x_j) \vee \lim_i d(x_j, y_i)$, for all j.

(3) $\overline{\lim_i}\, d(x_i, y_i) \leq \underline{\lim_j}\, (\lim_i d(x_i, x_j) \vee \lim_i d(x_j, y_i)) \leq$
$\underline{\lim_j}\, \lim_i d(x_j, y_i) \vee \overline{\lim_j}\, \lim_i d(x_i, x_j)$.

(4) $\overline{\lim_i}\, d(x_i, y_i) \leq \underline{\lim_j}\, \lim_i d(x_j, y_i)$.

(5) $d(x_j, y_i) \leq d(x_i, y_i) \vee d(x_i, x_j)$, for all i, j.

(6) $\lim_i d(x_j, y_i) \leq \underline{\lim_i}\, d(x_i, y_i) \vee \lim_i d(x_i, x_j)$, for all j.

(7) $\underline{\lim_j}\, \lim_i d(x_j, y_i) \leq \underline{\lim_i}\, d(x_i, y_i)$.

(4) and (7) now yield.

(8) $\overline{\lim_i}\, d(x_i, y_i) \leq \underline{\lim_i}\, d(x_i, y_i)$, which completes the proof.

Let S_1 denote the class of all Cauchy sequences in S and define for $\{x_i\}$, $\{y_i\}$ in S_1, $\{x_i\} \sim \{y_i\}$ if and only if $d(\{x_i\}, \{y_i\}) = \lim_i d(x_i, y_i) = 0$.

<u>Lemma 4</u>. <u>The relation $\sim$ is an equivalence relation in S_1</u>.

<u>Proof</u>. The relation $\sim$ is clearly reflexive and symmetric. To show transitivity, let $\{x_i\} \sim \{y_i\}$ and $\{y_i\} \sim \{z_i\}$. Then

(1) $d(x_i, z_i) \leq d(x_i, y_i) \vee d(y_i, z_i)$, for all i.

Thus,

(2) $\overline{\lim_i}\, d(x_i, z_i) \leq \overline{\lim_i}\, (d(x_i, y_i) \vee d(y_i, z_i)) = \lim_i d(x_i, y_i) \vee \lim_i d(y_i, z_i) = 0$. This concludes the proof.

We denote by S_2 the set of equivalence classes of S_1 under $\sim$. That is $S_2 = S_1/\sim$. If $\{x_i\} \in S_1$, we denote by $[\{x_i\}]$ the member of S_2 containing $\{x_i\}$. We define in S_2, $d([\{x_i\}], [\{y_i\}]) = d(\{x_i\}, \{y_i\})$. It is clear that S_2 will be a S-metric space under this distance function provided the distance function is independent of the members of equivalence classes chosen to compute it.

<u>Lemma 5</u>. <u>If, in</u> S_1, $\{x_i\} \sim \{y_i\}$ <u>and</u> $\{w_i\} \sim \{z_i\}$, <u>then</u> $d(\{x_i\}, \{w_i\}) = d(\{y_i\}, \{z_i\})$.

<u>Proof</u>.

(1) $d(x_i, w_i) \leq d(x_i, y_i) \vee d(y_i, z_i) \vee d(z_i, w_i)$, for all i.

Thus,

(2) $\lim_i d(x_i, w_i) \leq \lim_i d(x_i, y_i) \vee \lim_i d(y_i, z_i) \vee \lim_i d(z_i, w_i) = \lim_i d(y_i, z_i)$.

Similarly,

(3) $\lim_i d(y_i, z_i) \leq \lim_i d(x_i, w_i)$.

We are now ready to show that S_2 is a completion for S of the Cantor-Hausdorff type (20).

<u>Theorem 5</u>. S_2 <u>is a complete S-metric space and</u> S <u>is congruent</u>

(12) <u>to a dense subset of</u> S_2

Proof. By the preceding lemmas and remarks, S_2 is a S-metric space. It is clear that S is congruently imbedded (12) in S_2 by the mapping $x \to \left[\{x_i\}\right]$, where $\{x_i\}$ denotes a sequence all of whose members are x. The density in S_2 of the image of S under this mapping follows in the usual fashion (20), by the selection of sequences diagonally, that is, if $\left[\{x_i\}\right]$ is an element of S_2 then $\left[\{x_i\}\right] = \text{d-}\lim_i \left[\{x_j^i\}\right]$ where $x_j^i = x_i$; for all j, for all i. Thus we need show only that S_2 is complete. Lemma 1 and the lemma of this section are used without specific mention. Let $\left\{\left[\{x_j^i\}\right] j\right\}$ i be a Cauchy sequence in S_2. Then,

(1) $\lim_i \lim_j \lim_k d(x_k^i, x_k^j) = 0$

From (1), there is a $b(j)$ so that $k(j) \geq b(j)$ implies,

(2) $\lim_i \lim_j d(x_{k(j)}^i, x_{k(j)}^j) = 0.$

Also,

(3) $\lim_j \lim_k d(x_j^i, x_k^i) = 0$; for all i.

From (3), there is an $a(i)$ so that $j(i) \geq a(i)$ implies

(4) $\lim_i \lim_k d(x_{j(i)}^i, x_k^i) = 0.$

Let $j(i) \geq \max(a(i), b(i))$. Then $\lim_v d(x_{j(i)}^i, x_{j(v)}^i)$ exists by (4) and a subsequence argument, and

(5) $\lim_i \lim_v d(x_{j(i)}^i, x_{j(v)}^i) = 0.$

Also, $\lim_v d(x_{j(v)}^i, x_{j(v)}^v)$ exists by (2) and a subsequence argument, and

(5') $\lim_i \lim_v d(x_{j(v)}^i, x_{j(v)}^v) = 0.$

Now,

(6) $d(x^i_{j(i)}, x^v_{j(v)}) \leq d(x^i_{j(v)}, x^v_{j(v)}) \vee d(x^i_{j(i)}, x^i_{j(v)})$,

for all i, v.

Thus,

(7) $\overline{\lim_v}\, d(x^i_{j(i)}, x^v_{j(v)}) \leq \lim_v d(x^i_{j(v)}, x^v_{j(v)}) \vee \lim_v d(x^i_{j(i)}, x^i_{j(v)})$

for all i.

Now, by a rather complicated sequence of applications of the triangle inequality and limiting processes, one shows that $\lim_v d(x^i_{j(i)}, x^v_{j(v)})$ exists.

It then follows from (5), (5'), and (7) that

(8) $\lim_i \lim_v d(x^i_{j(i)}, x^v_{j(v)}) = 0$ so that $\{x^i_{j(i)}\}$ is a Cauchy sequence. Thus $\left[\{x^i_{j(i)}\}\right] \in S_2$. Finally, by the triangle inequality,

(9) $\lim_k d(x^i_k, x^k_{j(k)}) \leq \lim_k d(x^i_k, x^i_{j(i)}) \vee \lim_k d(x^i_{j(i)}, x^k_{j(k)})$;

for all i. From (5), (5'), and (9), then, we have

(10) $\lim_i \lim_k d(x^i_k, x^k_{j(k)}) = 0$ so that

(11) $d\text{-}\lim_i \left[\{x^i_k\}\right] = \left[\{x^k_{j(k)}\}\right]$ and we have shown that S_2 is complete.

5. Digression. An application.

A filtre, F, is a system of subsets of a set, E, satisfying the following conditions.

(1) The null set (0) does not belong to F.

(2) if $u \in F$, $u \subset v$, then $v \in F$.

(3) If $u \in F$, $v \in F$, then $u \cap v \in F$.

A base, B, for a filtre, F, is a system of subsets satisfying the following conditions

(4) The null set does not belong to B.

(5) If $u \in B$, $v \in B$, then there exists a $w \in F$, such that $u \cap v \subset w$.

By EE we mean the set of ordered couples (x, y), where $x, y \in E$. The diagonal of EE is the set of couples (x, x), where $x \in E$.

A filtre, F, defined on EE, defines a uniform structure if it satisfies the following conditions.

(6) Any element of the filtre contains the diagonal.

(7) If we designate by u^{-1} the set of (y, x), where $(x, y) \in u$, then $u^{-1} \in F$ if $u \in F$.

(8) If we designate by u^2 the set of (x, y) such that there exists a z with $(x, z) \in u$, $(z, y) \in u$, then there is a $u \in F$ so that $u^2 \subset v$ if $v \in F$.

The filtre F is said to be the filtre of the uniform structure. If $u \in F$ then $u(a)$ is the set of x's such that $(a, x) \in u$.

An element, u, of the filtre is idempotent if $u^2 = u$ and is symmetric if $u^{-1} = u$.

A topological space which is T_1 is zero-dimensional if and only if it has a base for open sets consisting of open and closed sets.

Theorem 9. Let S be a space of uniform structure defined by a filtre having a symmetric, idempotent base. Then, S is zero-dimensional.

Proof. Let F be the filtre of the hypothesis and B the base. Let G be any open set in S and p any point of G. Then, since G contains a neighborhood of each of its points, there is $v \in F$ with $v(p) \subset G$. Since B is a base for F, there are $u_1, u_2, \ldots, u_n \in B$ with $\bigcap_{i=1}^{n} u_i \subset v$. Thus, $\bigcap_{i=1}^{n} u_i(p) \subset v(p)$. To show that $\bigcap_{i=1}^{n} u_i(p) = u(p)$ is open and closed, it will suffice to show that $u_i(p)$ is open and closed for $i = 1, 2, \ldots, n$. Since u_i is symmetric and idempotent, $(x, y) \in u_i$ is an equivalence relation. Thus, the neighborhoods defined in S by the n_i are a disjoint covering of S. These are open since they are neighborhoods and closed since their complements are open.

Theorem 10. If S is a zero-dimensional space, the topology of S is compatible with a uniform structure defined by a filtre having a symmetric, idempotent base closed under finite intersection.

Proof. Let S be zero-dimensional. Then, there exist coverings of S by disjoint open sets. Let $C = \{G_\alpha\}$ be any such covering. Define $u_C = \{(x, y) \mid \exists \alpha \ni x \in G_\alpha \text{ and } y \in G_\alpha\}$. Clearly, each u_C is symmetric and idempotent. Also, the intersection of u_C and $u_{C'}$ is a $u_{C''}$, for if $C = \{G_\alpha\}$,

$G' = \{G_\beta^*\}$, and $G'' = \{G_\alpha \cap G_\beta^*\}$, then $u_{G''} = u_G \cap u_{G'}$. Thus, no finite intersection of the u_G is null, and the minimal dual ideal (3) in 2^{SS} containing u_G is a filtre F of the prescribed type since each u_G contains the diagonal of SS and if $v \supset \bigcap_{i=1}^{n} u_{G_i}$, then $v \supset (\bigcap_{i=1}^{n} u_{G_i})^2$. Now, if G is any open set in S containing a point p, then G contains an open and closed set, G^*, containing p and G^* with its complement is an open and closed covering of S in which the neighborhood $u_G(p)$ of p is G^*. Thus, every neighborhood of p in the topology of S contains a neighborhood of p in the topology induced by F. Let $v \in F$. Then there are u_{G_i} in the base of F so that $v \supset \bigcap_{i=1}^{n} u_{G_i}$. But, $\bigcap_{i=1}^{n} u_{G_i}(p)$ is an open set in the topology of S and contains p. Thus, every neighborhood of p in the topology induced by F contains a neighborhood of p in the topology of S and the two topologies are equivalent.

Combining Theorems 9 and 10 we obtain

<u>Theorem 11</u>. <u>A topological space is zero-dimensional if and only if its topology is compatible with a uniform structure defined by a filtre having a symmetric, idempotent base closed under finite intersection</u>.

<u>Proof</u>. An intersection of idempotents is idempotent and we may close a base under finite intersection without altering the induced topology.

Theorem 12. Let S be a space of uniform structure defined by a filtre F having a symmetric, idempotent base closed under finite intersection. Then S is metrizable over the Boolean algebra $_2 2^{SS}$.

Proof. Let B be the base of F described in the hypotheses. Define $B(x, y) = \{u \in B \mid (x, y) \in u\}$ and define $d(x, y) = B - B(x, y)$. Since the diagonal of SS is in all elements of F, $d(x, y)$ satisfies the vanishing condition one way. S is, however, understood to be T_1 so that $d(x, y) = 0$ implies $x = y$. The symmetry of $d(x, y)$ is immediate from the symmetry of B. Now let $u \in d(x, y)$. If $u \notin d(x, z)$ and $u \notin d(y, z)$, then $(x, y) \in u^2 = u$, which is a contradiction. Hence, the triangle inequality is satisfied and S forms a $_2 2^{SS}$ metric space under $d(x, y)$.

Let $\lim_n x_n = x$ in the topology of S. Suppose also that there is an element u in B so that $u \in \overline{\lim_j}\, d(x, x_j)$. Then for a given integer k there is $j \geq k$ so that $(x, x_j) \notin u$ which is impossible since u contains almost all (x_j, x). Thus, the topology of S is stronger than the induced distance topology.

Suppose next that $\overline{\lim_j}\, d(x, x_j) = 0$ and there is a neighborhood, $u(x)$, which excludes infinitely many x_j. (This follows from a neighborhood $v(x)$ general, in F, since B is a base of F and is closed under finite intersection). Then $u \in d(x, x_j)$ for infinitely many x_j, which is a contradiction.

Thus, the induced distance topology is stronger than the topology of S and the theorem is proved.

Combining the last two theorems we may state

<u>Theorem 13</u>. <u>If S is a zero-dimensional space then S is metrizable over</u> $_2 2^{SS}$.

6. <u>Weak topological products</u>.

Let S_n be a sequence of B-metric spaces. Denote the combinatory product by $\prod_n S_n$ and the projection of $\prod_n S_n$ onto S_n by πn: $\prod_n S_n \to S_n$. Define a distance function, and hence a topology, in $\prod_n S_n$ by $d(x, y) = \bigvee_{i=1}^{n} d(x_i, y_i)$.

<u>Lemma 6</u>. <u>For each</u> n, πn: $\prod_n S_n \to S_n$ <u>is continuous</u>.

<u>Proof</u>. If $\lim_m d(x^m, x) = 0$, then $\lim_m d(x_n^m, x_n) = 0$.

<u>Lemma 7</u>. <u>For each</u> n, πn. $\prod_n S_n \to S_n$ <u>is open</u>.

<u>Proof</u>. Let C be an open set in $\prod_n S_n$. Suppose there is a sequence x_n^m in the complement of πn (C) and a point x_n in πn (C) with $\lim_m d(x_n^m, x_n) = 0$ for some given n. Select x any point of C whose n^{th} coördinate is x_n. Take x^m with m^{th} coördinate x_n^m and the remaining coördinates those of x. Then $\lim_m d(x^m, x) = 0$ and the complement of C is not closed, which is a contradiction.

Combining Lemmas 6 and 7 we have

<u>Theorem 14</u>. $\prod_n S_n$ <u>forms a B-metric space under</u> $d(x, y) = \bigvee_{i=1}^{n} d(x_i, y_i)$. <u>The topology induced by this distance function is a weak product topology in</u> $\prod_n S_n$ (25).

In a fashion similar to that of Lemmas 6 and 7, one may show that if S_k is a B_k-metric space for a collection (S_k) and if B is the cardinal product of B_k, then $\prod_k S_k$ forms a B-metric space under $d(x, y) = (d_1(x_1, y_1), d_2(x_2, y_2), \ldots)$ and the induced distance topology is again a weak product topology.

It is an open question as to under what conditions the weak product topologies just mentioned will be the corresponding strong product topologies. This question remains open even after introducing a sequence of "convergence factors" in the distance function as is done in the metrization of a product of metric spaces.

7. <u>A special case. Set algebras in the autometrization.</u>

In general, infinite set algebras may be shown to be dense in themselves. An open question is the density-in-self of general σ-complete Boolean algebras.

A second such question is the compactness of subsets of general algebras. We define a subset of a set algebra to be bounded provided the distance set (in the autometrization) of the subset has as elements only countable sets. A set is finitely compact if its bounded subsets are compact.

<u>Theorem 15</u> <u>Set algebras are finitely compact.</u>

<u>Proof</u>. Select any denumerably infinite bounded subset. It suffices to show that $\lim_i x_i = x$ for some subsequence

$\{x_i\}$ of the bounded subset (considered well-ordered). If the bounded subset (considered well-ordered) has a limit, the theorem is proved. Suppose, however, that $\overline{\lim_i}\, x_i = y$ contains $\underline{\lim_i}\, x_i = z$ as a proper subset. If $y - z$ is a finite set $(w_1, w_2, \ldots w_n)$ then we delete all of the elements in $\{x_i\}$ which contain w_1 and all of the elements in the remaining sequence which contain w_2 and so on through w_n. The remaining sequence will have a limit. If $y - z$ is a countable set $(w_1, w_2, \ldots w_n, \ldots)$ then we delete all of the elements in $\{x_i\}$ which contain w_1 and all of the elements in the remaining set which contain w_2 and so on. If there is a sequence left after denumerably many deletions then the resulting sequence has a limit. If there is no sequence left then consider the first element in $\{x_i\}$ which was rejected because it contained w_1 and the first element which was rejected because it contained w_2 and so on. This yields a sequence $\{x_{i_n}\}$ which has a limit. Suppose $y - z$ is an uncountable set, then let x_k be an element of $\{x_i\}$ which contains an uncountable number of elements of $y - z$. It follows that the cardinal of $(x_k \cap y \cap z') \geq \aleph_1$ (also the cardinal of $x_k \cap e' \geq \aleph_1$).

Thus,

(1) $w_i = d(x_k, x_i)$; where w_i is countable; for each i (by hypothesis).

(2) $\overline{\lim_i}\, w_i = \overline{\lim_i}\, ((x_k' \cap x_i) \cup (x_k \cap x_i')) \geq (x_k' \cap \overline{\lim_i}\, x_i) \cup (x_k \cap (\underline{\lim_i}\, x_i)') = (x_k' \cap y) \cup (x_k \cap z')$.

But $\overline{\lim_i}\, w_i$ has cardinal $\leq \aleph_0$, and since $x_k \cap a'$ has cardinal $\geq \aleph_1$ so does $(x_k' \cap y) \cup (x_k \cap z')$. This is a contradiction.

<u>Corollary</u>. <u>The set algebra of a countable set is compact</u>.

Let $\{A_n\}$ be a sequence in the set algebra of the set algebra of the positive integers where A_n is the set of all sets of positive integers which have n as a member. No subsequence of $\{A_n\}$ is convergent. This example, suggested by Mr. Alfred Lehman, shows that general set algebras need not be compact.

5. <u>A conjecture</u>.

The attempt has been made, without success, to find an algebraic hold on closed sets in B and, hence, in B-metric spaces. This problem is close to Problem 76 of Birkhoff (3). One might suspect that the closed intervals form a sub-base for closed sets in B. This, however, is apparently not the case.

<u>Conjecture</u>. <u>The σ-sublattices of B form a sub-base for closed sets in B</u>.

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BIOGRAPHICAL SKETCH

H D Sprinkle was born June 18, 1926 in Logansport, Indiana. He attended Culver Military Academy and Edison High School. After graduating from high school he entered the United States Navy. He attended Georgia Institute of Technology, the University of Miami, and the University of Arizona where he received the Bachelor of Arts degree in 1949. He took the Master of Arts degree in 1950 from the University of Miami where he held a graduate assistantship after attending a summer at the University of Florida. He then went to India to study Oriental Philosophy. Upon returning from India he entered the University of Florida and there completed the work for the Doctor of Philosophy degree in 1953. During his stay at the University of Florida he was a graduate assistant and interim-instructor. He is a member of Phi Beta Kappa, Phi Kappa Phi, Pi Mu Epsilon, the American Mathematical Society, and the Association for Symbolic Logic.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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